UTILIZATION OF SIMILARITY CONSIDERATIONS FOR THE IMPROVEMENT OF THE CONVERGENCE OF A PROCESS OF SUCCESSIVE APPROXIMATION IN SHELL ANALYSIS

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This paper represents an extension of [1]; the idea of the work is connected, on the one hand, with the studies of the Chinese scientist, Chien, [2] in which the deflections of circular plates are determined by means of expansions of the stresses in powers of the deflection of the center of the plate, and, on the other hand, with the work of Mushtari [3] on the semi-nonlinear treatment of the problem of the determination of deflections of shells. In [3] the nonlinear equations are linearized with respect to the higher harmonics of the deflection in stress functions, and the non-linearity with respect to the fundamental harmonic of the deflection having the largest amplitude is taken into account. This permits the inclusion of the principal part of the nonlinearity connected with the large deflections and considerably simplifies the calculations.

At each step of the consecutive approximations presented in this work the equations are likewise found to be comparatively satisfactory with respect to the fundamental harmonics of the larger amplitudes; the determination of the higher harmonics of the deflections and the stress function is reduced to the solution of linear equations. Since the higher harmonics are usually small and their frequencies high, in determining them the curvature of the shell and the nonlinearities of the problems have been disregarded.

1. The equations of the theory of shallow shells may be written in abbreviated form [4]:

$$\triangle^2\Phi = Eh\left\{\tfrac{1}{2}[w, w] + [w, w^\circ]\right\} \tag{1.1}$$

$$D\Delta^2 w + [w^\circ, \Phi] + [w, \Phi] = pP \qquad \left(D = \frac{Eh^3}{12(1-\nu^2)}\right) \tag{1.2}$$

using the notation

$$[w, \Phi] = 2w_{xy}\Phi_{xy} - w_{yy}\Phi_{xx} - w_{xx}\Phi_{yy}, \qquad [w, w] = 2(w_{xy}{}^2 - w_{xx}w_{yy})$$
$$w_{xx} = \partial^2 w/\partial x^2, \qquad \Delta^2 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy}$$

Here E is the modulus of elasticity, h the thickness of the shell, ν Poisson's ratio, w° the height of the shell above the foundation before deformation, w the deflection, p the amplitude of the loading of the shell, P the loading function, Φ the stress function in terms of which the membrane stresses T_x, T_y, T_{zy} are readily expressed as well as the strains of the middle surface ϵ_x, ϵ_y, ϵ_{xy}; for example,

$$T_x = \Phi_{yy}, \qquad \varepsilon_x = \frac{1}{Eh}(\Phi_{yy} - \nu\Phi_{xx}) = u_{1x} + \frac{1}{2}w_x{}^2 + w_x w_x{}^0 \tag{1.3}$$

where u_1, u_2 are the displacement components in the tangent planes to the shell.

It is easy to obtain the required similarity theorem [4,5]. Introduce the notation

$$v^\circ = w^\circ/h, \quad v = w/h, \quad u_1{}^* = u_1/h^2 \quad \Phi^* = \Phi/Eh^3, \quad u_2{}^* = u_2/h^2$$

where v° is the reduced height of the shell above the base, v the reduced deflection. Dividing (1.1) and (1.3) by Eh^3, (1.2) by Eh^4 and (1.4) by h^2, one obtains

$$\Delta^2\Phi^* = \frac{1}{2}[v, v] + [v, v^\circ]$$
$$\frac{1}{12(1-\nu^2)}\Delta^2 v + [v^\circ, \Phi^*] + [v, \Phi^*] = \frac{p}{Eh^4}P$$
$$T_x{}^* = T_x/Eh^3 = \Phi_{yy}{}^*, \qquad \Phi_{yy}{}^* - \nu\Phi_{xx}{}^* = u_{1x}{}^* + \frac{1}{2}v_x{}^2 + v_x v_x{}^\circ$$

Using these relations, it may be deduced that two shells having the same relative height v° above the bases, differing only by the modulus of elasticity and thickness and having the same relative geometric dimensions will experience the identical reduced deflections $v = w/h$, provided the reduced transverse loads p/Eh^4 and the reduced tangential displacements of the edges of the shell u_1/h^2 and u_2/h^2 are identical and the longitudinal edge loads are related to each other as Eh^3.

2. Taking into consideration the above similarity theorem, one may make the deduction that in order to obtain the relationship between the reduced loads p/Eh^4 and the relative deflections w/h it is sufficient to study a shell of any arbitrary thickness and any modulus of elasticity.

Using this fact to improve the rate of convergence of the successive approximations, at each step of the approximations one will not only change the deflections but also the loading, the thickness and the modulus of elasticity of the shells and their dimensions in the same manner in order to obtain more exact satisfaction of the equations.

All the geometrical dimensions, except for the thickness of the shell, will be left unaltered.

3. Description of the method. Consider a shallow shell, freely supported or clamped along the edges and under transverse loading. Using the relation $w^0 = hv^0$, Equations (1.1) and (1.2) will be reduced to the form

$$\Delta^2\Phi = Eh\left\{\tfrac{1}{2}[w, w] + h[v^\circ, w]\right\}$$
$$\Delta^2 w = D^{-1}\{-h[v^\circ, \Phi] - [w, \Phi] + pP\} \tag{3.1}$$

Substituting $h = \sqrt{12(1-\nu^2)D/Eh}$, one obtains

$$\Delta^2\Phi = Eh\left\{\tfrac{1}{2}[w, w] + \sqrt{12(1-\nu^2)D/Eh}\ [v^0, w]\right\} \tag{3.2}$$

$$\Delta^2 w = D^{-1}\{-[w, \Phi] + pP - \sqrt{12(1-\nu^2)D/Eh}\ [v^\circ, \Phi]\} \tag{3.3}$$

Substitute now the arbitrary values $Eh = (Eh)_0$ and some approximate expression for the deflection $w = w_0$ of the shell in the original approximation and determine from the equation

$$\Delta^2\Phi_0 = (Eh)_0\left\{\tfrac{1}{2}[w_0, w_0] + \sqrt{12(1-\nu^2)D/(Eh)_0}\,[v^\circ, w_0]\right\} \tag{3.4}$$

the function Φ_0, satisfying the support conditions of the shell. Substituting the found value into (3.3), solve for w_n with $n = 1$ the equation

$$\Delta^2 w_n = D^{-1}\{-[w_{n-1}, \Phi_{n-1}] + P_n p - \sqrt{12(1-\nu^2)D/(Eh)_{n-1}}\ [v^\circ, \Phi_{n-1}]\} \tag{3.5}$$

Its solution will be presented in the form $w_n = w_n' + p_n w_n''$, where w_n'' is that part of the deflection which depends on the external loading p_n.

This quantity will be determined from the condition of equality in the nth and $(n+1)$th approximations of a certain generalized displacement (w, ϕ); the symbol (w, ϕ) denotes the scalar product of the deflection function into some approximately selected function ϕ, considered as vectors of a functional space. In the capacity of such a generalized displacement it is convenient to choose the generalized displacement for which the rigidity of the shell is smallest. Since the smallest rigidity is obviously attached only to one of the orthogonal, between themselves, generalized displacements, it follows that if the generalized displacements of small rigidities are equal, the total displacements will differ but little from each other.

However, as will be shown later, the convergence will be sufficiently good even in the case when the function ϕ does not correspond to minimum rigidity of the shell. For example, in order to simplify the calculations, one may take in the capacity of the function ϕ the Dirac function and determine the quantity p_n from the condition of equality of the deflection of the center of the shell a in two consecutive approximations

$$w_n(a) = w_n'(a) + p_n w_n''(a) = w_{n-1}(a), \quad p_n = \{w_{n-1}(a) - w_n'(a)\} / w_n''(a) \tag{3.6}$$

The following approximations for Φ are obtained by means of the solution for $n = 1$ of the equations

$$\Delta^2 \Phi_n = (Eh)_n \left\{ \tfrac{1}{2} [w_n, w_n] + \sqrt{12(1-\nu^2) D / (Eh)_n} \, [v^0, w_n] \right\} \tag{3.7}$$

The solution will depend on the rigidity in extension $(Eh)_n$. This last quantity will be determined so as to fulfil the condition

$$(\Phi_n, \varphi) = (\Phi_{n-1}, \psi) \tag{3.8}$$

for some function ψ. Using the relations (3.9)

$$h_n = \sqrt{12(1-\nu^2) D / (Eh)_n}, \qquad E_n = (Eh)_n / h_n \qquad (D = E_0 h_0^3 / 12[1-\nu^2])$$

the corresponding values of the thickness h_n and modulus of elasticity E_n of the shell in the nth approximation will be determined, where for this purpose the cylindrical rigidity $E_n h_n^2 (1 - \nu^2)$ will be the same for all approximations.

Since the solution of the equation (3.7) is determined exactly only within a linear function of the coordinates which does not influence the magnitude of the membrane forces (membrane forces are determined by second derivatives, these derivatives of linear functions are equal to zero), the function ψ will be chosen orthogonal to the linear functions

$$(1, \psi) = 0, \qquad (x, \psi) = 0, \qquad (y, \psi) = 0 \tag{3.10}$$

As a result of the above process of successive approximations one may obtain sufficiently exact corresponding values of the deflections, stresses and stress functions for shells of a certain thickness (the thickness of the shell and its modulus of elasticity are determined in the process of calculation). Changing the original approximation w_0, one may establish a continuous set of values, corresponding to each other, of reduced deflections and loads. Analogously, one obtains the subsequent approximations.

In order to clarify the above study, consider the solution of the equation

$$\Delta^2 w_n = f_n \tag{3.11}$$

by the method of Fourier; f_n denotes here the right-hand side of Equation (3.5). Expand the function f_n in eigen functions of the operator Δ^2, i.e. in functions ϕ_i satisfying the boundary conditions of the support of the shell and the equations

$$\Delta^2\varphi_i = \lambda_i\varphi_i \quad (i = 1,2,\ldots). \qquad f_n = \sum_1^\infty f_{ni}\,\varphi_i \tag{3.12}$$

The solution of (3.11) will be sought in the form

$$w_n = \sum_1^\infty w_{ni}\,\varphi_i \tag{3.13}$$

Substituting (3.13) and f_n from (3.12) in (3.11), one finds, taking the first relation (3.12) into consideration,

$$\sum_1^\infty w_{ni}\,\lambda_i\varphi_i = \sum_1^\infty f_{ni}\,\varphi_i \tag{3.14}$$

Comparing coefficients of ϕ_i on both sides of this equation one obtains

$$w_{ni}\,\lambda_i = f_{ni}\,, \qquad w_{ni} = \frac{f_{ni}}{\lambda_i}\,, \qquad w_n = \sum_1^\infty \frac{f_{ni}}{\lambda_i}\,\varphi_i \tag{3.15}$$

Obviously, the eigen values of the operator Δ^2 form a rapidly increaing sequence; hence the magnitude λ_1 of the first eigen value will be considerably smaller in absolute value than the remaining eigen values λ_2, λ_3.

Therefore, the first term of the sum (3.15) will often be larger in magnitude than the sum of the remaining terms of the series. In order to achieve a greater rate of convergence it is desirable to change the load at each approximation step such that the coefficients f_{n1} remain unchanged, the other coefficients being small so that they will change but little. Under these conditions the consecutive approximations will be close to each other, and this will mean rapid convergence of the approximations.

Since the exact values of the eigen functions ϕ_i are unknown, one may regulate the loads for the execution of the calculations, in order to keep constant the quantities

$$J_n = (w_n \varphi) = \sum_1^{\infty} \frac{f_{ni}}{\lambda_i} (\varphi_i \varphi) \qquad (3.16)$$

Since the first eigen value is often a quantity which is much smaller than the remaining values, one has the approximate equality

$$(w_n, \varphi) = \sum_{i=1}^{\infty} \frac{f_{ni}}{\lambda_i} (\varphi_i, \varphi) \approx \frac{f_{n1}}{\lambda_i} (\varphi_i, \varphi) \qquad (3.17)$$

On the strength of this approximate equation the equality of the quantity $f_{n,i} = f_{(n-1),i}$ may be approximately replaced by the equality of the scalar products $(w_n, \phi) = (w_{n-1}, \phi)$.

This justifies the proposed procedure of the correction of the loading. It should be noted that the equality (3.17) will be the more exact the closer the function ϕ is to the first eigen function. In the capacity of this function one may recommend the use of the function w_0 which has as fundamental component the first eigen function ϕ_1; however, if this is too difficult for practical calculations, then this recommendation may not be maintained.

(*Note.* In an analogous manner one may correct at each step of approximation the thickness of the shell h in such a manner that there will be the possibility of keeping unchanged the component of the expansion of the stress function in eigen functions corresponding to the least non-zero eigen value (a linear function of coordinates is an eigen function of the zero eigen value of the operator Δ^2).

It may be expected that the above method of successive approximation will give faster converging consecutive approximations and its region of convergence will be wider than those of the earlier known methods. This may be expected, since for the above method at each approximation step the displacement components and stress function components of small rigidity remain unchanged, and only the components of the deflection and the stress functions for the higher eigen functions of larger rigidity change, which generally speaking are small in value; therefore, it may be expected that their changes at each approximation step will likewise be small, i.e. the process will be fast converging.

Obviously, the advantage of the described method over the previous methods of successive approximation is only evident in those cases where the first non-zero eigen value of the operator Δ^2 is considerably less than those of the remaining eigen values.)

4. Application of the method of successive approximations may be somewhat simplified by a modification which gives it the form of a method of expansion in series of powers of a small parameters.

Let Equations (3.2) and (3.3) be presented in the form

$$\Delta^2\Phi = \mu Eh\left\{\tfrac{1}{2}[w, w] + \sqrt{12(1-\nu^2)D/Eh}\ [v^\circ, w]\right\} \tag{4.1}$$

$$\Delta^2 w = \mu D^{-1}\{-[w, \Phi] - \sqrt{12(1-\nu^2)D/Eh}[v^\circ, \Phi]\} + D^{-1}pP \tag{4.2}$$

The quantities w, Φ, Eh will now be sought in the form of the power series

$$p = p_0 + \mu p_1 + \mu^2 p_2 + \ldots, \qquad w = w_0 + \mu w_1 + \mu^2 w_2 + \ldots,$$
$$\Phi = \mu\Phi_1 + \mu^2\Phi_2 + \ldots, \qquad Eh = B_0 + \mu B_1 + \mu^2 B_2 + \ldots \tag{4.3}$$

where B_0 is an arbitrary positive number.

Substituting these expressions in (4.1) and (4.2) and comparing coefficients of the Maclaurin expansions in powers of μ on the left- and right-hand sides of the equations, one may obtain a series of recurrence relations for the successive determination of the coefficients in the series (4.3).

In addition to these relations, one must also use the equations

$$(w_0 + \mu w_1 + \ldots \mu^{n-1} w_{n-1}, \varphi) = (w_0 + \mu w_1 + \ldots + \mu^{n-1} w_{n-1} + \mu^n w_n, \varphi)$$
$$(\mu\Phi_1 + \ldots + \mu^{n-1}\Phi_{n-1}, \psi) = (\mu\Phi_1 + \ldots + \mu^{n-1}\Phi_{n-1} + \mu^n\Phi_n, \psi)$$

i.e. the equations

$$(w_n, \varphi) = 0 \ \text{ for } n > 0 \qquad (\varphi_n, \psi) = 0 \ \text{ for } n > 1$$

which are employed for the determination of the coefficients p_i and B_i. An example of the application of the method of expansion in a small parameter follows in the next section.

5. *Example 1.* In order to compare different versions of the method of successive approximations let us consider one of the few problems which have an exact solution, namely, the problem of Bubnov on the cylindrical bending of a plate of infinite length in one direction with its long edges supported in such a way that these edges may not move, but can freely rotate. The origin of coordinates will be placed at the middle of the plate and the x-axis parallel to the short side of the plate.

The membrane force T_x, arising in the plate, is determined by its relative extension [4]

$$T_x = \frac{Eh}{(1-\nu^2)l}\int_{-l}^{l}\frac{w_x^2}{2}dx, \quad T_y = \nu T_x \tag{5.1}$$

This force is the same at all points of the plate. The stress function has the form

$$\Phi = T_x \frac{1}{2} y^2 + \nu T_x \frac{1}{2} x^2 \tag{5.2}$$

The equilibrium equations and boundary conditions are known to have the form

$$Dw_{xxxx} = p + T_x w_{xx}, \qquad w(l) = w(-l) = w_{xx}(l) = w_{xx}(-l) = 0 \tag{5.3}$$

As an initial approximation for the deflection the deflection function will be chosen which is obtained from the elementary theory of infinitesimal deflections of plates, not taking into account membrane stresses:

$$w_0(x) = w_0\left[\left(\frac{x}{l}\right)^4 - 6\left(\frac{x}{l}\right)^2 + 5\right] \tag{5.4}$$

Giving E_0 and h_0 arbitrary values, approximate values of the membrane forces T_{x0} will be calculated with the aid of (5.1):

$$T_{x0} = \frac{(Eh)_0}{(1-\nu^2)\,l}\int_{-l}^{l}\frac{w_{0x}{}^2}{2}\,dx = \frac{15{,}54\,(Eh)_0 w_0{}^2}{(1-\nu^2)\,l^3} \qquad ((Eh)_0 = E_0 h_0) \tag{5.5}$$

These values are then substituted on the right-hand side of the first equation (5.3), which is then solved for the boundary conditions (5.3) for w. One obtains

$$Dw_{1xxxx} = p + T_{x0} w_{0xx}, \qquad D = E_0 h_0{}^3/(1-\nu^2) \tag{5.6}$$

$$w_1(x) = l^2 D^{-1} T_{x0} w_0\left[\frac{1}{30}\left(\frac{x}{l}\right)^6 - \frac{1}{2}\left(\frac{x}{l}\right)^4 + \frac{5}{2}\left(\frac{x}{l}\right)^2 - \frac{61}{30}\right] + \frac{1}{24}D^{-1}l^4 p_1\left[\left(\frac{x}{l}\right)^4 - 6\left(\frac{x}{l}\right)^2 + 5\right] \tag{5.7}$$

The magnitude of the load p_1 is now determined from the condition of equality of the deflections of the middle surface of the plate in the initial and first approximations $w_1(0) = w_2(0)$:

$$-\frac{61}{30} l^2 D^{-1} T_{x0} w_0 + \frac{5}{24} D^{-1} l^4 p_1 = 5w_0, \qquad p_1 = 24D\,(w_0/l^4) + 9.76 T_{x0}\,(w_0/l_2) \tag{5.8}$$

Substituting this expression in (5.7), one finds the correction $\delta_1(x)$ to the initial approximation $w_0(x)$

$$w_1(x) = w_0(x) + \delta_1(x), \qquad \delta_1(x) = D^{-1} T_{x0} l^2 w_0\left[\frac{1}{3}\left(\frac{x}{l}\right)^6 - \frac{7}{75}\left(\frac{x}{l}\right)^4 + \frac{3}{50}\left(\frac{x}{l}\right)^2\right] \tag{5.9}$$

Further, putting in (5.1) $Eh = (Eh)_1$, and evaluating with the aid of (5.5) and (5.10) the new approximation for the membrane forces, one finds

$$T_{x1}=\frac{(Eh)_1}{(1-\nu^2)\,l}\int\limits_{-l}^{l}\frac{1}{2}w_{1x}{}^2dx=\frac{(Eh)_1}{(1-\nu^2)\,l}\left\{\int\limits_{-l}^{l}\frac{1}{2}w_{ax}{}^2dx+\int\limits_{-l}^{l}w_{0x}(x)\,\delta_{1x}dx+\int\limits_{-l}^{l}\frac{1}{2}\delta_{1x}{}^2dx\right\} \tag{5.10}$$

$$T_{x1}=\frac{(Eh)_1}{(1-\nu^2)}\{0.000345\,(l^2D^{-1}T_{x0})^2+0.04464l^2D^{-1}T_{x0}+15.54\}\,w_0{}^2 \tag{5.11}$$

If in correspondence with the ideas of the method of expansion in powers of a small parameter one neglects in (5.10) the integral of the square of the small correction $\delta_x{}^2$ in comparison with other terms as a quantity of higher order of smallness, one obtains the formula

$$T_{x1}=\frac{(Eh)_1}{1-\nu^2}\{0.04464l^2D^{-1}T_{x0}+15.54\}\,w_0{}^2 \tag{5.12}$$

In accordance with the formula (5.2) the values of the stress functions in the initial and first approximations are equal, respectively to the quantities

$$\Phi_0=T_{x0}\left(\frac{1}{2}y^2+\frac{1}{2}\nu x^2\right),\qquad \Phi_1=T_{x1}\left(\frac{1}{2}y^2+\frac{1}{2}\nu x^2\right)$$

In the present case Equation (3.8) reduces to

$$T_{x1}\left(\left[\frac{1}{2}y^2+\frac{1}{2}\nu x^2\right],\psi\right)=T_{x0}\left(\left[\frac{1}{2}y^2+\frac{1}{2}\nu x^2\right],\psi\right)\quad \text{или}\quad T_{x1}=T_{x0} \tag{5.13}$$

Substituting in it the last relation (5.12) and (5.15), one finds the equation for the determination of $(Eh)_1$:

$$\frac{(Eh)_1}{(1-\nu^2)\,l^2}\,(0.4464l^2D^{-1}T_{x0}+15.54)\,w_0{}^2=\frac{(Eh)_0}{(1-\nu^2)\,l^2}\,15.54w_0{}^2$$

Solving this equation for $(Eh)_1$ and replacing T_{x0} by its expression (5.5), one has

$$(Eh)_1=(Eh)_0\left[1+\frac{0.4874}{1-\nu^2}\left(\frac{w_0}{h_0}\right)^2\right]^{-1} \tag{5.14}$$

Further, from (3.9), one finds the thickness and the modulus of elasticity of the plate in the second approximation

$$h_1=\sqrt{12\,(1-\nu^2)\,D/(Eh)_0},\qquad E_1=(Eh)_1/h_1 \tag{5.15}$$

Using (5.5), (5.7) and (5.15), one may by use of a series of increasing values of w_0 evaluate the corresponding approximations of the deflections, loads, membrane forces, thickness of the plate and their moduli of elasticity. Thus, the solution of the problem is not only obtained for one specified plate, but for different loadings for different plates; however, this is completely sufficient for the purpose of construction of graphs of the dependence between the reduced load p_1/E_1h^4, the reduced deflection w_1 and the reduced membrane stresses T_1/E_1h^3.

Using the preceding relations one may deduce the following approximate relations between the deflection of the middle of the plate and its load:

$$\frac{p_1}{E_1 h_1^4} = 0.4395\,\frac{w_1(0)}{h'} + 1.3335\,\frac{w_1(0)^3/h_1^3}{1 - 0.02143\,(w_1(0)/h_1)^2} \tag{5.16}$$

However, the use of these formulas is not at all necessary, as one may with equal success construct the graphs of the relations between the reduced deflection and the reduced loading as this has been indicated above.

In the following table are given the results of computations of the values of the reduced loads $p^\circ = pl^4/Eh^4$, causing different relative deflections of the middle of the plate; in the first column are stated the relative deflections of the middle of the plate, in the second the corresponding exact values of the reduced stresses, in the third and fourth the approximate values of the loading evaluated by the method of Chien [2] by means of expansion of the loading in powers of the deflections in the second and third approximations, in the fifth column are found the values of the loading, evaluated in the second approximation by the described method with the aid of Formula (5.16).

Next to the approximate values are given the percentages of their relative errors.

TABLE

w/h	p° exact	p°[1] 2nd approximation	p°[1] 3rd approximation	p° (5.16) 2nd approximation
0.365(4)	0.225(9)	0.225(7), 0.1%	0.226, 0.05%	0.225(8), 0.05%
0.728	0.843	0.835, 1	0.845(6), 0.4	0.841(4), 0.25
1.087	2.24	2.19, 2.7	2.20, 0.9	2.23, 0.5
1.44	4.8(0)	4.6(4), 3.3	4.9(5). 3.1	4.8(3), 0.6
1.7(8)	8.9(3)	8.4(0), 6	9.3(0), 4.5	8.9(6), 0.(4)
2.1(4)	15.(0)	14.2, 6	16.4, 9	15.6, 4
2.5(6)	23.(4)	22(0), 6	26(9), 17	25(3), 8
3.2(2)	48.(8)	46(0), 6	63(2), 3(0)	58(9), 2(0)

6. *Example 2.* Let us now consider the problem of the determination of the deflection of a circular plate of radius a which is subjected to a uniformly distributed pressure p. The edge of the plate is assumed to be rigidly clamped; this means that the rotations and displacements of the edge of the plate are assumed to vanish. Let r denote the distance from the center of the plate, w the deflection, N the magnitude of the radial

membrane stress, E the modulus of elasticity, ν Poisson's ratio and h the thickness of the plate.

As has been shown in [2], introducing the new coordinate

$$x = 1 - (r^2/a^2) \tag{6.1}$$

allows us to give the compatibility equation and the corresponding boundary condition the form

$$[(1-x),\ N]_{xx} = -\tfrac{1}{2} Ehw_x{}^2 \qquad 2N_x - (1-\nu)N = 0 \quad \text{for} \quad x = 0 \tag{6.2}$$

The magnitude of the radial membrane force N is related to the stress function Φ by

$$N = \Phi_r/r = -2\Phi_x/a^2 \tag{6.3}$$

The equilibrium equation and boundary conditions for the deflection have the form (cf. [1])

$$-[(1-x)\,w_x]_{xx} = \frac{3(1-\nu^2)}{4}\frac{a^4 p}{Eh^3} - 3(1-\nu^2)\,Nw_x, \quad w = w_{1x|} = 0 \quad \text{for} \quad x = 0 \tag{6.4}$$

As initial approximation for the form of the deflection, the deflection shape of the plate for infinitesimal loading will be taken: it is determined by (6.4), if one sets there $N = 0$;

$$w_0(x) = w_0 x^2 \tag{6.5}$$

Substituting this value $w_0(x)$ for w on the right-hand side of (6.2), replacing E and h by arbitrary positive numbers E_0 and h_0 and solving the corresponding equation for N, one finds

$$N_0 = E_0 h_0 w_0{}^2 \frac{1}{6}\left[\frac{2}{1-\nu} + x + x^2 + x^3\right] \tag{6.6}$$

The corresponding values of the stress function are determined by means of the solution of the equation [cf. (6.3)] $2\Phi_{0x}/a^2 = N_0$.

The solution of this equation has the form

$$\Phi_0(x) = -\frac{1}{2} a^2 \int_0^x N_0 dx + \Phi_0(0) \tag{6.7}$$

$$\Phi_0(x) = -E_0 h_0 a^2 \frac{1}{12}\left(\frac{2}{1-\nu}x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4\right) + c$$

Substituting (6.5) and (6.6) for w and N on the right-hand side of (6.4) the resultant equation will be solved for w_1; the magnitude of the load $p = p_1$ in the first approximation will be chosen such that the deflections of the center of the plate in the first and initial approximations coincide; as a result of the computations one finds for the deflection

$$w_1(x) = w_0(x) + \delta_1 w(x) \tag{6.8}$$

$$\delta_1 w(x) = \frac{w_0^3}{h^2}(1-\nu^2)x^2(1-x)\left[\frac{83-43\nu}{1-\nu} + 23x + 8x^2 + 2x^3\right] \tag{6.9}$$

for the load

$$p_1 = \frac{16}{3}\frac{E_0 h_0^3}{a^4(1-\nu^2)}\left\{w_0 + \frac{1+\nu}{360}(173-73\nu)\frac{w_0^3}{h_0^2}\right\} \tag{6.10}$$

Further, according to (3.4), the quantity Φ_1 will be determined by means of solution of the equation

$$[(1-x)N_1]_{xx} = -\frac{1}{2}E_1 h_1 w_1^2, \qquad 2N_{1x} - (1-\nu)N_1 = 0 \quad \text{for} \quad x = 0 \tag{6.11}$$

and the successive determination of Φ from the equations [cf. (6.3)]

$$-\frac{2\Phi_{1x}}{a^2} = N_1, \qquad \Phi_1(x) = -\frac{a^2}{2}\int_0^x N_1 dx + \Phi_1(0) \tag{6.12}$$

If one substitutes in (6.11) the expressions (6.8) and (6.9) and in correspondence with the method of a small parameter neglects the square of a small correction and its products, one may obtain from (6.12)

$$\begin{aligned}\Phi_1 = -\frac{1}{2}a^2E_1h_1\Bigg\{&\frac{1}{6}\left(\frac{2}{1-\nu}x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4\right)w_0^2 + \\ &+ \frac{(1-\nu^2)}{h^2 7560}w_0^4\left[\frac{160-104\nu}{(1-\nu)^2}x + \frac{80-52\nu}{1-\nu}\left(\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4\right) - \right. \\ &\left. - \frac{501-249\nu}{5(1-\nu)}x^5 - \frac{123}{6}x^6 - \frac{39}{7}x^7 - \frac{9}{8}x^8\right]\Bigg\} + \Phi_1(0)\end{aligned} \tag{6.13}$$

The quantity $E_1 h_1$ will be determined from the condition of equality of the mean values of the face N in the initial and first approximations

$$\frac{1}{a}\int_0^a N_0 dx = \frac{1}{a}\int_0^a N_1 dx$$

In accordance with (6.7) and (6.12) this equation reduces to the condition

$$-\frac{1}{2a^3}\{\Phi_0(a) - \Phi_0(0)\} = -\frac{1}{2a^3}\{\Phi_1(a) - \Phi_1(0)\} \tag{6.14}$$

which corresponds to (3.8) for

$$\psi(x) = -\frac{1}{2a^3}\{\delta(x-a) - \delta(x)\}$$

where $\delta(x)$ is the Dirac-Dirichlet function.

If one substitutes in (6.14) the earlier expression for the stress function, one finds for $\nu = 0.3$

$$0.657E_0h_0w_0^2 = E_1h_1\left(0.657w_0^2 + 0.0257\frac{w_0^4}{h_0^2}\right) \tag{6.15}$$

Since it has been agreed to change the quantities E_1h in such a manner that for this the cylindrical rigidity does not change

$$E_0h_0^3 = E_1h_1^3, \ \ldots \tag{6.16}$$

Thus, dividing the corresponding terms of (6.15) by the terms of (6.16) one obtains

$$\frac{0.657w_0^2}{h_0^2} = \frac{1}{h_1^2}\left(0.657w_0^2 + 0.0257\frac{w_0^4}{h_0^2}\right), \qquad h_0^2 = h_1^2 - 0.0392w_0^2 \tag{6.17}$$

Replacing in accordance with (6.16) in (6.10) $E_0h_0^3$ by $E_1h_1^3$ and expressing in the same formula the quantity h_0^2 by h_1^2 from (6.17), one may obtain for $\nu = 0.3$

$$\frac{p_1}{E_1h_1^4} = \frac{5.86}{a^4}\left\{\frac{w_0}{h_1} + 0.544\left(\frac{w_0}{h_1}\right)^3\left[1 - 0.0392\left(\frac{w_0}{h_1}\right)^2\right]^{-1}\right\} \tag{6.18}$$

7. For cases of supports of shells for which the equations of the linear theory of shells are easily solved, one may apply a faster converging method of successive approximation.

For this purpose let $\Phi^* = \Phi\sqrt{Eh}$ and reduce the system of equations (3.2) and (3.3) by means of transfer of all linear terms to the left-hand side to the form

$$\Delta^2\Phi^* - \sqrt{12(1-\nu^2)D}\,[v,\ w] = \tfrac{1}{2}\sqrt{Eh}\,[w,\ w] \tag{7.1}$$

$$D\Delta^2 w + \sqrt{12(1-\nu^2)D}\,[v,\ \Phi^*] = P - \sqrt{Eh}\,[w,\ \Phi^*] \tag{7.2}$$

The system of equations will now be solved successively

$$\Delta^2\Phi^*_{n-1} - \sqrt{12(1-\nu^2)D}\,[v,\ w_{n+1}] = \tfrac{1}{2}\sqrt{(Eh)_{n-1}}\,[w_n,\ w_n] \tag{7.3}$$

$$D\Delta^2 w_{n-1} + \sqrt{12(1-\nu^2)D}\,[v,\ \Phi^*_{n+1}] = p_{n+1}P - \sqrt{Eh_{n+1}}\,[w_n,\ \Phi^*_n] \tag{7.4}$$

For this purpose the quantities $(Eh)_{n+1}$ and p_{n+1} at each step of approximation are adjusted so that the quantities

$$(\Phi^*_{n+1},\ \psi) = (\Phi^*_n,\ \psi), \qquad (w_{n+1},\ \varphi) = (w_n,\ \varphi)$$

remain unchanged.

The above method may be applied without change to the determination of the deflections of shells under longitudinal edge loadings. Application of the method of perturbation and of the method of successive approximations to longitudinal bending without utilization of the similarity theorem for the improvement of the convergence has been given in the papers [6,7].

BIBLIOGRAPHY

1. Svirskii, I.V., Razlichnye varianty metoda posledovatel'nykh priblizhenii i metoda vozmushchenii (Different forms of the method of successive approximations and the method of perturbations). *Izv. Kazansk. Fil. AN SSSR* No. 12, pp. 29-41, 1948.

2. Chien, W.Z., Large deflection of a circular clamped plate under uniform pressure. *Chinese J. of Phys.* Vol. 7, No. 2, 1947.

3. Mushtari, Kh.M., Srednii progib pologoi obolochki, priamougol'noi v plane i opiraiushcheisia na gibkie v svoei ploskosti rebra (Mean bending of shallow shells of rectangular plan form and supported on a flexible rib in its plane). *Izv. Kazansk. fil. AN SSSR Ser. fiz. mat. i tekhn. nauk* No. 12, pp. 53-62, 1958.

4. Mushtari, Kh.M. and Talimov, K.Z., *Nelineinaia teoriia uprugikh obolochek* (*Nonlinear theory of elastic shells*). Tatknigoizdat, 1957.

5. Vol'mir, A.S., *Gibkie plastinki i obolochki* (*Flexible shells and plates*). GITTL, 1956.

6. Polubarinova-Kochina, P.Ia., K voprosu ob ustoichivosti plastin (On the problem of the stability of a plate). *PMM* Vol. 20, No. 1, 1956.

7. Alekseev, S.A., Poslekriticheskaia rabota gibkich uprugikh plastinok (Postbuckling behavior of flexible plates). *PMM* Vol. 20, No. 6, pp. 673-679, 1956.

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